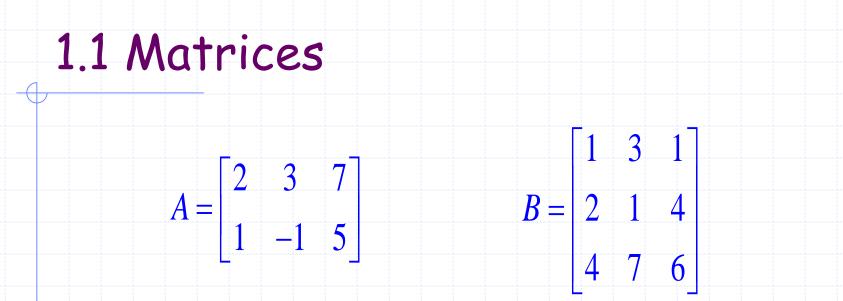
MATRICES AND DETERMINANTS

Dr. NITAKSHI GOYAL Assistant Professor in Mathematics, Akal Degree College, Mastuana Sahib Punjab, India.

- 1.2 Operations of matrices
- 1.3 Types of matrices
- 1.4 Properties of matrices
- 1.5 Determinants
- 1.6 Inverse of a 3×3 matrix



Both A and B are examples of matrix. A matrix is a rectangular array of numbers enclosed by a pair of bracket.

Why matrix?

Consider the following set of equations:

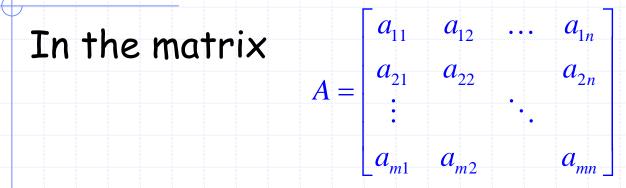
 $\begin{cases} x + y = 7, & \text{It is easy to show that } x = 3 \text{ and} \\ 3x - y = 5. & y = 4. \end{cases}$

$$x + y - 2z = 7$$

How about solving $\begin{cases} 2x - y - 4z = 2, \\ -5x + 4y + 10z = 1, \\ 3x - y - 6z = 5. \end{cases}$

Matrices can help...

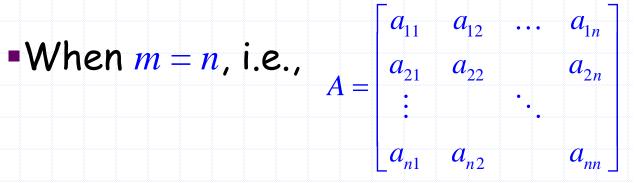




•numbers a_{ij} are called <u>elements</u>. First subscript indicates the row; second subscript indicates the column. The matrix consists of <u>mn</u> elements

•It is called "the $m \times n$ matrix $A = [a_{ij}]$ " or simply "the matrix A" if number of rows and columns are understood.

Square matrices



- A is called a "square matrix of order n" or "n-square matrix"
- elements a_{11} , a_{22} , a_{33} ,..., a_{nn} called diagonal elements.

• $\sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \dots + a_{nn}$ is called the *trace* of *A*.

Equal matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal (A = B) iff each element of A is equal to the corresponding element of B, i.e., $a_{ij} = b_{ij}$ for $1 \le i \le m, 1 \le j \le n$.

•iff pronouns "if and only if"

if A = B, it implies $a_{ij} = b_{ij}$ for $1 \le i \le m, 1 \le j \le n$;

if $a_{ij} = b_{ij}$ for $1 \le i \le m, 1 \le j \le n$, it implies A = B.

Equal matrices

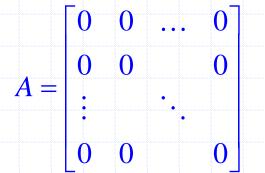
Example: $A = \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Given that A = B, find a, b, c and d.

if A = B, then a = 1, b = 0, c = -4 and d = 2.

Zero matrices

Every element of a matrix is zero, it is called a zero matrix, i.e.,



1.2 Operations of matrices Sums of matrices •If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices, then A + B is defined as a matrix C = A + B, where $C = [c_{ij}], c_{ij} = a_{ij} + b_{ij}$ for $1 \le i \le m, 1 \le j \le n$. **Example:** if $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{bmatrix}$ Evaluate A + B and A - B. $A+B = \begin{vmatrix} 1+2 & 2+3 & 3+0 \\ 0+(-1) & 1+2 & 4+5 \end{vmatrix} = \begin{vmatrix} 3 & 5 & 3 \\ -1 & 3 & 9 \end{vmatrix}$ $A - B = \begin{vmatrix} 1 - 2 & 2 - 3 & 3 - 0 \\ 0 - (-1) & 1 - 2 & 4 - 5 \end{vmatrix} = \begin{bmatrix} -1 & -1 & 3 \\ 1 & -1 & -1 \end{vmatrix}$ 10

Sums of matrices

•Two matrices of the <u>same</u> order are said to be *conformable* for addition or subtraction.

 Two matrices of <u>different</u> orders cannot be added or subtracted, e.g.,

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 $\begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ 4 & 7 & 6 \end{bmatrix}$

are NOT conformable for addition or subtraction.

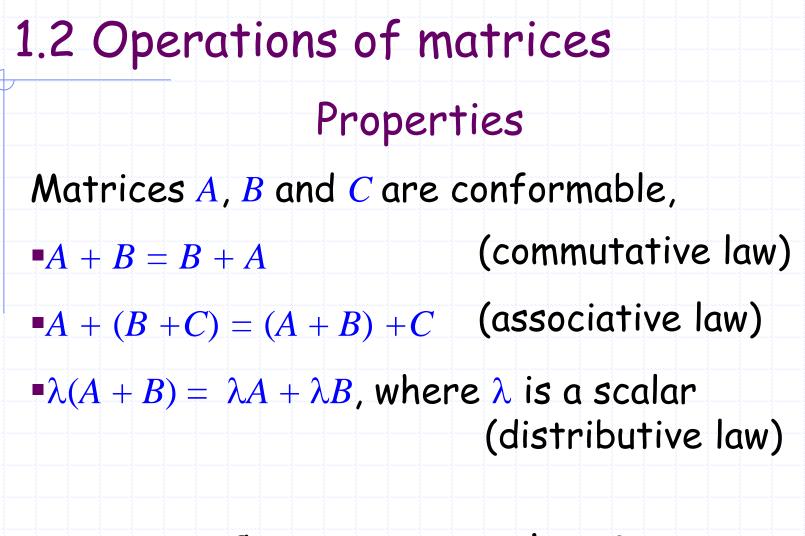
Scalar multiplication

•Let λ be any scalar and $A = [a_{ij}]$ is an $m \times n$ matrix. Then $\lambda A = [\lambda a_{ij}]$ for $1 \le i \le m, 1 \le j \le n$, i.e., each element in A is multiplied by λ .

Example: $A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{vmatrix}$. Evaluate 3A.

 $3A = \begin{bmatrix} 3 \times 1 & 3 \times 2 & 3 \times 3 \\ 3 \times 0 & 3 \times 1 & 3 \times 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 3 & 12 \end{bmatrix}$

•In particular, $\lambda = -1$, i.e., $-A = [-a_{ij}]$. It's called the *negative* of A. Note: A - A = 0 is a zero matrix



Can you prove them?

Properties

Example: Prove $\lambda(A + B) = \lambda A + \lambda B$.

Let
$$C = A + B$$
, so $c_{ij} = a_{ij} + b_{ij}$.

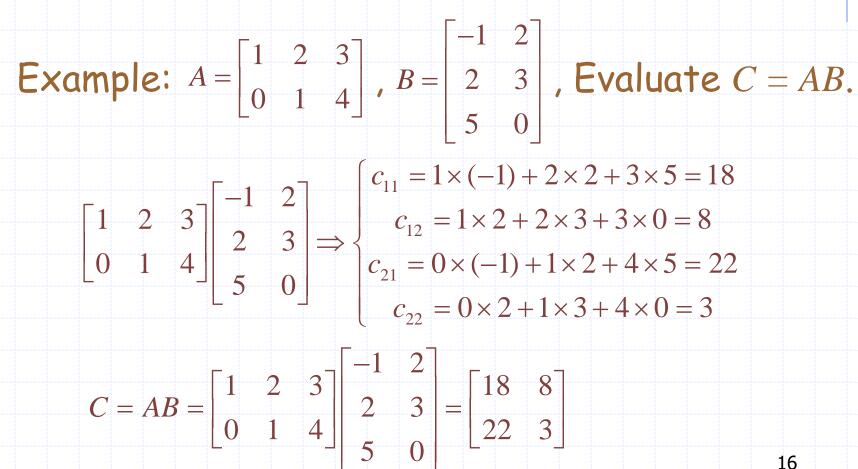
Consider $\lambda c_{ij} = \lambda (a_{ij} + b_{ij}) = \lambda a_{ij} + \lambda b_{ij}$, we have, $\lambda C = \lambda A + \lambda B$.

Since $\lambda C = \lambda (A + B)$, so $\lambda (A + B) = \lambda A + \lambda B$

Matrix multiplication

•If $A = [a_{ij}]$ is a $m \times p$ matrix and $B = [b_{ij}]$ is a $p \times n$ matrix, then AB is defined as a $m \times n$ matrix C = AB, where $C = [c_{ii}]$ with $c_{ij} = \sum_{ik}^{r} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{ip} b_{pj}$ for $1 \le i \le m, \ 1 \le j \le n$. Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix}$ and C = AB. Evaluate c_{21} . $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{vmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{vmatrix} \quad c_{21} = 0 \times (-1) + 1 \times 2 + 4 \times 5 = 22$ 15 15

Matrix multiplication



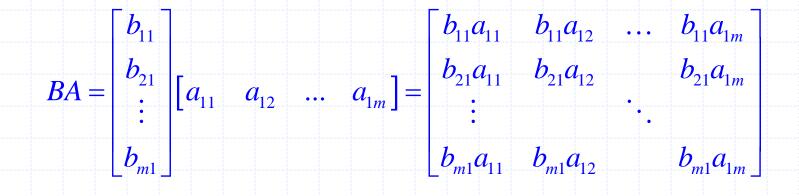
1.2 Operations of matrices **Matrix multiplication**

•In particular, A is a $1 \times m$ matrix and **B** is a $m \times 1$ matrix, i.e., $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix}$

then C = AB is a scalar. $C = \sum_{i=1}^{m} a_{1k}b_{k1} = a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1m}b_{m1}$

Matrix multiplication

BUT BA is a $m \times m$ matrix!



•So $AB \neq BA$ in general !

1.2 Operations of matrices Properties Matrices A, B and C are conformable, $\bullet A(B + C) = AB + AC$ $\bullet (A + B)C = AC + BC$ $\bullet A(BC) = (AB) C$ $AB \neq BA \text{ in general}$ AB = 0 NOT necessarily imply A = 0 or B = 0 AB = AC NOT necessarily imply B = C

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Properties

Example: Prove A(B + C) = AB + AC where A, B and C are *n*-square matrices

Let X = B + C, so $x_{ij} = b_{ij} + c_{ij}$. Let Y = AX, then

$$y_{ij} = \sum_{k=1}^{n} a_{ik} x_{kj} = \sum_{k=1}^{n} a_{ik} (b_{kj} + c_{kj})$$
$$= \sum_{k=1}^{n} (a_{ik} b_{kj} + a_{ik} c_{kj}) = \sum_{k=1}^{n} a_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} c_{kj}$$

So Y = AB + AC; therefore, A(B + C) = AB + AC

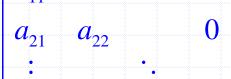
1.3 Types of matrices

- Identity matrix
- The inverse of a matrix
- The transpose of a matrix
- Symmetric matrix
- Orthogonal matrix

1.3 Types of matrices Identity matrix

•A square matrix whose elements $a_{ij} = 0$, for i > j is called upper triangular, i.e., $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \end{bmatrix}$

• A square matrix whose elements $a_{ij} = 0$, for i < j is called lower triangular, i.e., $a_{11} = 0$...



0



1.3 Types of matrices Identity matrix

- Both upper and lower triangular, i.e., $a_{ij} = 0$, for $i \neq j$, i.e., $D = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & a_{mn} \end{bmatrix}$
 - is called a diagonal matrix, simply
 - $D = \text{diag}[a_{11}, a_{22}, \dots, a_{nn}]$

1.3 Types of matrices Identity matrix •In particular, $a_{11} = a_{22} = ... = a_{nn} = 1$, the matrix is called identity matrix. •Properties: AI = IA = AExamples of identity matrices: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 1.3 Types of matrices Special square matrix

•*AB* \supseteq *BA* in general. However, if two square matrices *A* and *B* such that AB = BA, then *A* and *B* are said to be *commute*.

Can you suggest two matrices that must commute with a square matrix A?

.. , xintam ytitnsbi sht , flszti A : 2nA

•If A and B such that AB = -BA, then A and B are said to be *anti-commute*.

1.3 Types of matrices The inverse of a matrix

•If matrices A and B such that AB = BA = I, then B is called the inverse of A (symbol: A^{-1}); and A is called the inverse of B (symbol: B^{-1}).

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Show B is the the inverse of matrix A.

Ans: Note that $AB = BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ detaile?

details?

1.3 Types of matrices The transpose of a matrix

The matrix obtained by interchanging the rows and columns of a matrix A is called the transpose of A (write A^T).

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ The transpose of A is $A^{T} = \begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix}$

For a matrix $A = [a_{ij}]$, its transpose $A^T = [b_{ij}]$, where $b_{ij} = a_{ji}$.

1.3 Types of matrices Symmetric matrix

- •A matrix A such that $A^T = A$ is called symmetric, i.e., $a_{ji} = a_{ij}$ for all *i* and *j*.
- $A + A^T$ must be symmetric. Why?
- Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$ is symmetric.
- A matrix A such that $A^T = -A$ is called skewsymmetric, i.e., $a_{ji} = -a_{ij}$ for all *i* and *j*.

•A - A^T must be skew-symmetric. Why?

1.3 Types of matrices Orthogonal matrix

- A matrix A is called orthogonal if $AA^T = A^TA = I$, i.e., $A^T = A^{-1}$
- Example: prove that $A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$ is
- Since, $A^{T} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$. Hence, $AA^{T} = A^{T}A = I$. Can you show the details?
- We'll see that orthogonal matrix represents a rotation in fact! 29

1.4 Properties of matrix

• $(AB)^{-1} = B^{-1}A^{-1}$ • $(A^{T})^{T} = A$ and $(\lambda A)^{T} = \lambda A^{T}$ • $(A + B)^{T} = A^{T} + B^{T}$

 $\bullet (AB)^T = B^T A^T$

- 1.4 Properties of matrix
- **Example:** Prove $(AB)^{-1} = B^{-1}A^{-1}$.
- Since $(AB) (B^{-1}A^{-1}) = A(B B^{-1})A^{-1} = I$ and
- $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = I.$
- Therefore, $B^{-1}A^{-1}$ is the inverse of matrix AB.

1.5 Determinants Determinant of order 2 Consider a 2 × 2 matrix: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

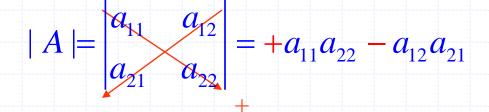
•Determinant of A, denoted |A|, is a <u>number</u> and can be evaluated by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

1.5 Determinants

Determinant of order 2

easy to remember (for order 2 only)..



Example: Evaluate the determinant: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

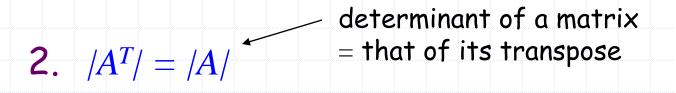
$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 3 = -2$$

1.5 Determinants

The following properties are true for determinants of <u>any</u> order.

1. If every element of a row (column) is zero,

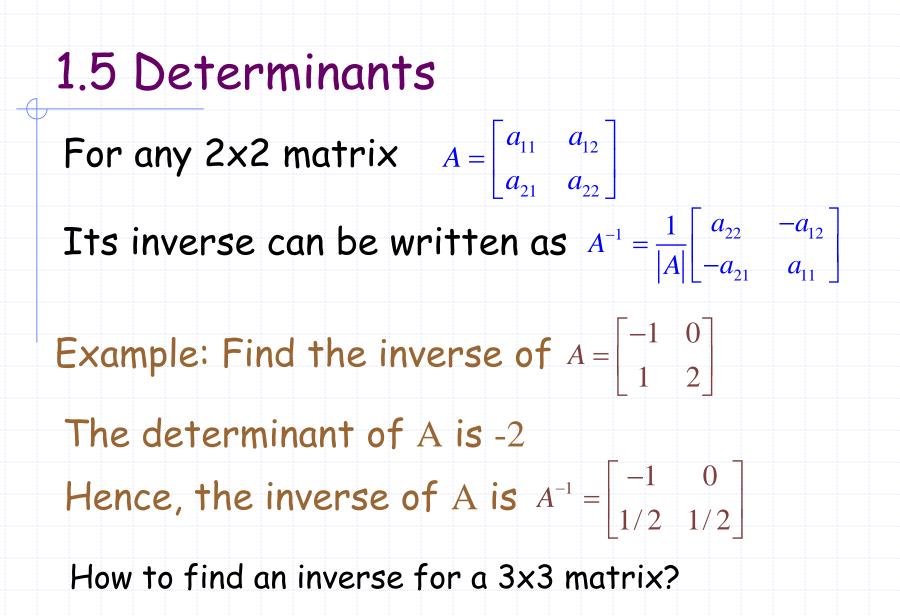
e.g., $\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 1 \times 0 - 2 \times 0 = 0$, then |A| = 0.



3. |AB| = |A|/|B|

1.5 Determinants

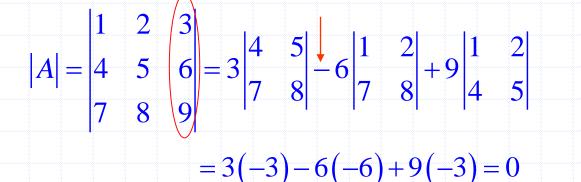
- Example: Show that the determinant of any orthogonal matrix is either +1 or -1.
- For any orthogonal matrix, $A A^T = I$.
- Since $|AA^T| = |A|/|A^T| = 1$ and $|A^T| = |A|$, so $|A|^2 = 1$ or $|A| = \pm 1$.



1.5 Determinants of order 3

Consider an example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Its determinant can be obtained by:

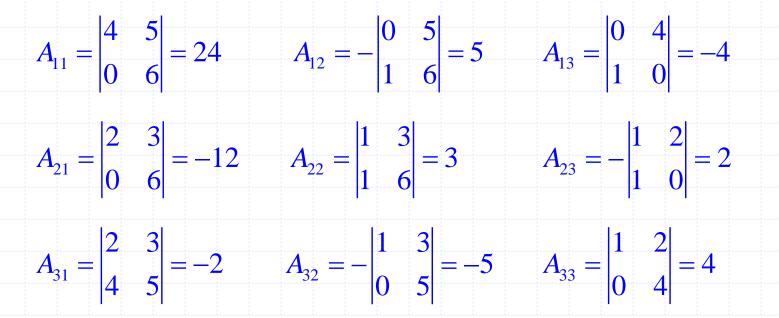


You are encouraged to find the determinant by using other rows or columns

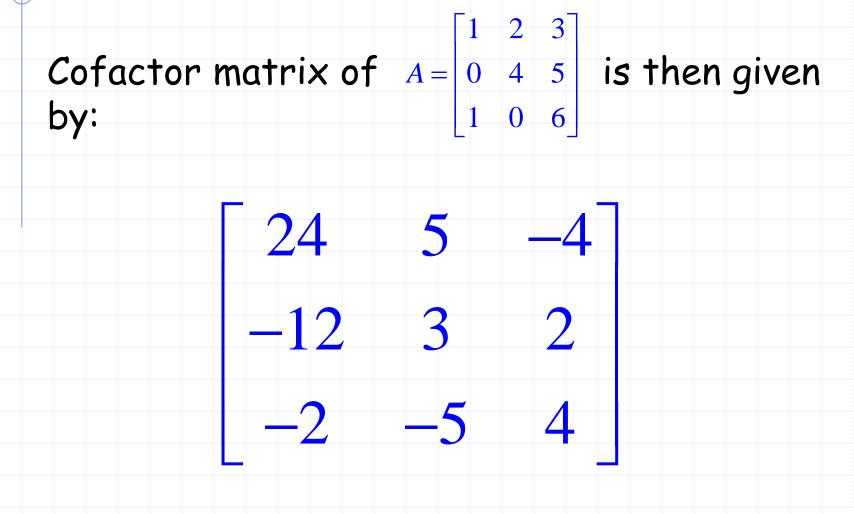
1.6 Inverse of a 3×3 matrix

Cofactor matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$

The cofactor for each element of matrix A:



1.6 Inverse of a 3×3 matrix



1.6 Inverse of a 3×3 matrix

